Oscillations and Waves - Lecture 8

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March 15, 2004

9.4 Fourier transform

- If the period of oscillation is allowed to become infinite, we can calculate the spectral density of any non-periodic function, e.g. a wave pulse.

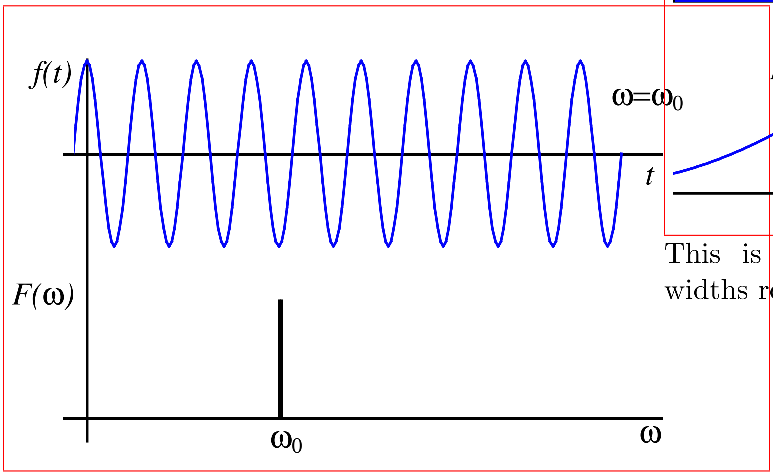
- In this limit, the Fourier coefficients become continuous functions i.e. all frequencies are allowed.

- The (infinite) array of Fourier coefficients is referred to as the *Fourier transform* of the function.

- Thus we can show that **any** arbitrary wave pulse can be produced from a superposition of harmonic waves, which is why we attach such importance to understanding their behaviour.

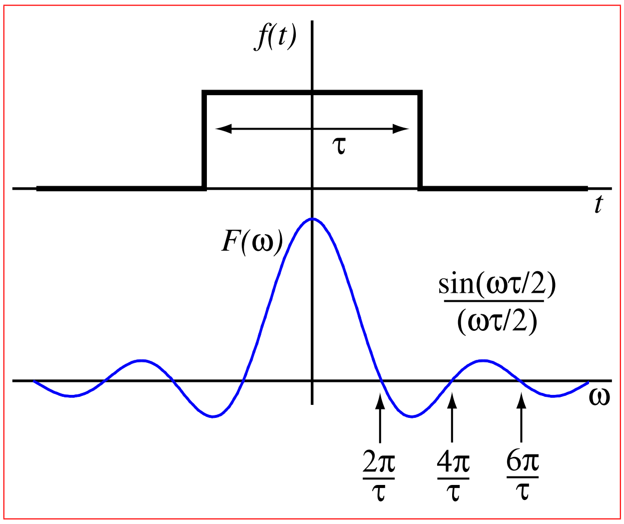
9.5 Some important Fourier transforms

9.5.1 Cosine wave:

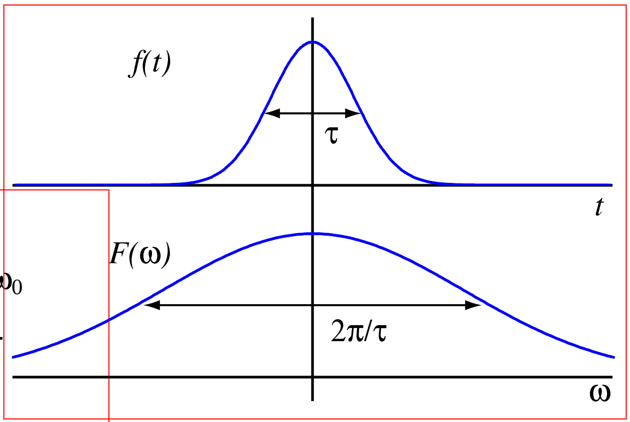


Actually, there is also a spike at $-\omega_0$, because the transform is symmetric.

9.5.2 Top hat:



9.5.3 Gaussian function:



This is just another Gaussian function, with widths related reciprocally, as shown.

9.6 The bandwidth theorem

(Tipler 16.5)

- The relationship between the widths of the two Gaussian functions shown above is important and has far reaching consequences.
- If the Gaussian function represents a wave pulse with width, in time, Δt, the Fourier transform tells us it contains a spread of frequencies, $\Delta\omega$.
- But, the product $\Delta t\Delta\omega$ is a constant, i.e.

$$\boxed{\Delta t\Delta\omega = 2\pi}$$

- This is known as the bandwidth theorem.
- A similar relationship is found if we look at the spatial width of the pulse:

$$\boxed{\Delta x\Delta k = 2\pi}$$

.

- For a more general pulse shape, the determination of the width of the pulse becomes more arbitrary, and so the bandwidth theorem becomes:

$$\boxed{\Delta t\Delta\omega \approx 2\pi \quad \text{and} \quad \Delta x\Delta k \approx 2\pi}$$

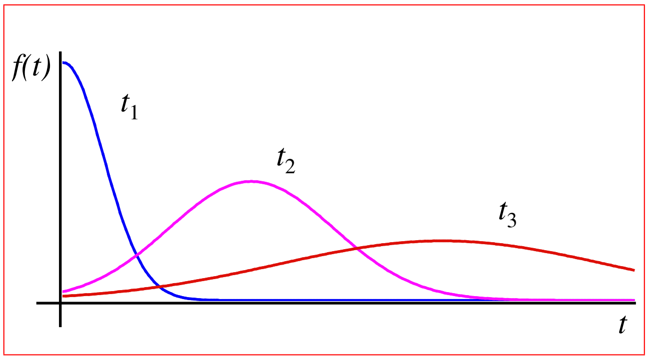
The implications of the bandwidth theorem are as follows:

- A **narrow** wave pulse will contain a **wide** range of frequencies;
- A **broad** wave pulse will contain a **narrow** range of frequencies;
- $\Delta\omega = 0$ implies $\Delta t = \infty$ i.e. an infinite sine wave, with a single frequency – a perfect wave;
- Similarly, $\Delta k = 0$ implies $\Delta x = \infty$.

10 Wave packets and dispersion

(Tipler 16.5)

- If all the frequency components in a wave packet or pulse travel at the same phase velocity, v_p, the resulting disturbance will propagate without changing shape, and the medium is known as **non-dispersive**.
- The velocity of propagation of the packet, is known as the **group velocity**, v_g.
- In a non-dispersive medium, $v_g = v_p$.
- The group velocity is the velocity at which energy is carried through the medium.
- If the medium is **dispersive**, different frequencies travel with different values of v_p, and the wave packet spreads as it travels.



10.1 Two waves with same amplitude, different frequency (beats)

Consider two waves with the same amplitude but differing in frequency. For convenience, assume that their phase difference is zero at $t = 0$, and consider their displacements at an arbitrary x coordinate (e.g. $x = 0$):

$$\boxed{y_1 = y_0 \sin\omega_1 t \quad \text{and} \quad y_2 = y_0 \sin\omega_2 t}$$

The sum of the two waves is given by:

$$\boxed{y = y_1 + y_2 = y_0 \sin\omega_1 t + y_0 \sin\omega_2 t}$$

Making use of

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

again gives:

$$y = 2y_0 \sin\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

i.e.

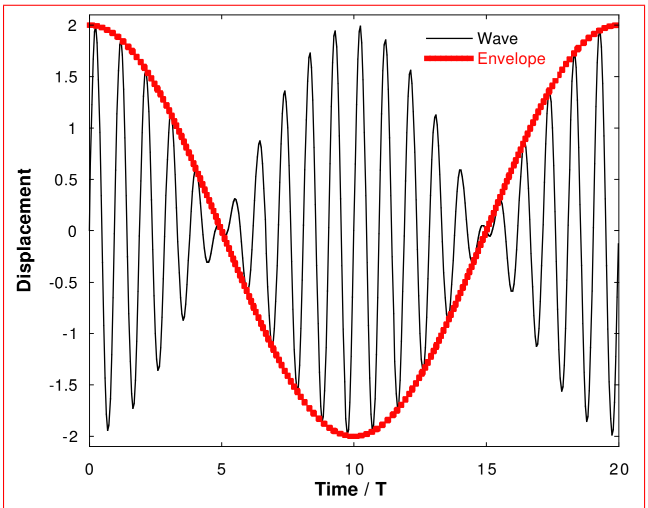
$$\boxed{y = \underbrace{2y_0\cos\left(\frac{\Delta\omega}{2}t\right)}_{\textit{slowly varying amplitude}} \underbrace{\sin\omega_{ave}t}_{\textit{wave with average frequency}}}$$

where $\Delta\omega = \omega_1 - \omega_2$.

The frequency of the resultant wave is the average of the two input waves, while the amplitude oscillates with frequency $\Delta\omega/2$.

This is known as beating.

The actual frequency of the beats is double this, as can be seen from the graph below.



i.e. the beat frequency = the difference in frequency between the two sources.

Repeating the analysis above, but with the full expressions for the two sine waves (i.e. $\sin(kx-\omega t)$ yields:

$$\boxed{y = 2y_0\cos(\frac{\Delta kx - \Delta\omega t}{2}).\sin(k_{ave}x - \omega_{ave}t)}$$

The phase velocity of the "average" wave is:

$$v_{ave} = \frac{\omega_{ave}}{k_{ave}}$$

whereas that of the envelope is:

$$v_{env} = \frac{\Delta\omega}{\Delta k}$$

In a non-dispersive medium, it is readily shown that $v_{ave} = v_{env} = \omega_1/k_1 = \omega_2/k_2 =$ the phase velocity of the medium.

However, in a dispersive medium, $v_{ave} \neq v_{env}$, and the "envelope" propagates at a different speed to the individual components. We associate v_{env} with the group velocity v_g, and in this particular case would write:

$$\boxed{v_g = \frac{\Delta\omega}{\Delta k}}$$

10.2 Dispersion

- In a dispersive medium, v_p is different for every frequency component, and $v_g \neq v_p$
- For any dispersive medium, we can write a relationship between ω and k, so that $\omega = \omega(k)$. This is called the dispersion relation, and depends on the physics of the particular wave phenomenon being observed.
- In the general case, the group velocity is given by the derivative of the dispersion relation, i.e.

$$\boxed{v_g = \frac{\partial\omega(k)}{\partial k}}$$

  (Proof of this result may be found in advanced textbooks – not Tipler).

- Usually, $v_g < v_p$ i.e. normal dispersion.
- However, can find $v_g > v_p$ i.e. the group, and hence the energy, travels faster than the individual waves. This is known as anomalous dispersion.
- Examples of dispersion:

- Splitting of light by a prism;
- Formation of a rainbow;
- Phonons propagating through a crystalline solid;
- Spreading of light pulses in fibre-optic cables, due to dispersion, limits the maximum length of cable before signal reconditioning is needed.